System of Equations

1. Solve: **(a)** $\begin{cases} x+y+z=4 \\ x+2y+z=2 \\ -x-2y+z=2 \end{cases}$ **(b)** $\begin{cases} 2x+4y+3z=4 \\ -x \qquad +5z=9 \\ 3x-5y-3z=2 \end{cases}$ **(c)** $\begin{cases} 2x+y=1 \\ 4x-6y=-6 \\ 2x-7y=-7 \end{cases}$

(d) $\begin{cases} x+3y-z=4 \\ x+2y+z=2 \\ 3x+7y+z=9 \end{cases}$ **(e)** $\begin{cases} x+3y-z=4 \\ x+2y+z=2 \\ 3x+7y+z=8 \end{cases}$

2. Find the constants α and β such that the system of equations $\begin{cases} x+y+3z=\alpha \\ 4x+\beta y-z=1 \\ 6x+7y+5z=2 \end{cases}$ has infinitely many solutions.

3. Consider the following system of linear equations (*) $\begin{cases} 4x+3y+z=\lambda x \\ 3x-4y+7z=\lambda y \\ x+7y-6z=\lambda z \end{cases}$.

Suppose λ is an integer and (*) has nontrivial solutions. Find λ and solve (*) .

4. **(a)** Find the value of the determinant $\begin{vmatrix} p+1 & 1 & 1 \\ 1 & p+1 & 1 \\ 1 & 1 & p+1 \end{vmatrix}$ where p is a real number.

(b) Consider the following system of equations:

$$(E) \quad \begin{cases} (p+1)x+y+z=p^2+3p \\ x+(p+1)y+z=p^3+3p^2 \\ x+y+(p+1)z=p^4+3p^3 \end{cases}.$$

(i) Find p such that (E) has a unique solution and find this solution.

(ii) Find p such that (E) has infinitely many solutions. In this case, solve (E).

(iii) Does there exist any p such that (E) has no solution? If your answer is 'yes', find it.

If your answer is 'no', give a brief explanation.

5. Find the condition for the equations: (E) $\begin{cases} (a-b-c)x+2ay+2a=0 \\ 2bx+(b-c-a)y+2b=0 \\ 2cx+2cy+(c-a-b)=0 \end{cases}$

to have a common solution and show that when this condition is satisfied, the equations have infinitely many common solutions.

6. **(a)** Let $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Evaluate $A^2 - 4A + 11\,I$. Hence find A^4 and A^{-1}.

(b) Solve the system of equations $\begin{cases} x+2y-3=0 \\ -4x+3y+5=0 \end{cases}$ by using matrix method.

7. Let $P = \begin{pmatrix} a & 1-a \\ 1-b & b \end{pmatrix}$, where a and b are real.

(a) Suppose $a + b \neq 2$. Show that there exists a unique real 1×2 matrix $\begin{pmatrix} x & y \end{pmatrix}$ satisfying

$$\begin{cases} x + y = 1 \\ \begin{pmatrix} x & y \end{pmatrix} P = \begin{pmatrix} x & y \end{pmatrix} \end{cases}.$$

(b) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} x & y \\ x & y \end{pmatrix}$, where $(x \quad y)$ is the unique 1×2 matrix described in **(a)** .

Show that there exists a real number λ such that $P - Q = \lambda(I - Q)$.

Prove that $P^n - Q = \lambda^n(I - Q)$ for all positive integer n.

8. **(a)** If $ab \neq 6$, find the inverse of $A = \begin{pmatrix} 0 & a & 1 \\ 2 & 5 & 0 \\ -2 & 1 & b \end{pmatrix}$.

(b) Discuss the solutions of the system of equations: (E) $\begin{cases} ay + z = 2 \\ 2x + 5y = 1 \\ -2x + y + bz = 3 \end{cases}$. Solve it in various cases.

9. Consider the following system of equations : $\begin{cases} x + y + 2z = 1 \\ 2x - y + 2z = 2 \\ 4x + y + 6z = c \end{cases}$, where c is real.

Suppose the system is consistent, find c and solve for x, y, z.

10. The system of equations : $\begin{cases} x + py + p^2z = 0 \\ x + qy + q^2z = 0 \\ x + ry + r^2z = 0 \\ x + y + z = 0 \end{cases}$ has the trivial solution

if and only if $x = y = z = 0$. By considering the coefficient determinants of any 3 of the given 4 equations, or otherwise, find the necessary and sufficient conditions that the system has non-trivial solution(s).

11. **(a)** Evaluate : $\begin{vmatrix} \alpha & 1 & 0 \\ 1 & 0 & \beta \\ 0 & \beta & \alpha \end{vmatrix}$, $\alpha \neq 0$, $\beta \neq -1$.

(b) Let $A = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & \beta \\ 0 & \beta & \alpha \end{pmatrix}$, $B = \begin{pmatrix} -\beta^2 & -\alpha & \beta \\ -\alpha & \alpha^2 & -\alpha\beta \\ \beta & -\alpha\beta & -1 \end{pmatrix}$, find AB and hence A^{-1}.

(c) Show that the system of equations : $\begin{cases} \alpha x + y = 1 \\ x + \beta z = 1 \\ \beta y + \alpha z = 1 \\ \alpha x + y - 2z = 1 \end{cases}$ has a unique solution if $\alpha\beta + 1 = \beta$.

12. **(a)** Find the multiplicative inverse, in terms of a, of the matrix with real numbers:

$$\begin{pmatrix} a+1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & a+1 \end{pmatrix} \quad , \quad \text{where } a \neq 0, -3.$$

(b) If a and b are real numbers, solve $\begin{cases} (a+1)x + y + z = 1 \\ x + (a+1)y + z = b \\ x + y + (a+1)z = b^2 \end{cases}$ in x, y and z in each of the cases:

(i) $a \neq 0, -3$ **(ii)** $a = 0$ **(iii)** $a = -3$

13. **(a)** Factorize the determinant $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & q^2 & r^2 \end{vmatrix}$.

(b) If p, q and r are all distinct, show that $(E):\begin{cases} x + y + z = p \\ px + qy + rz = pq \\ p^2x + q^2y + r^2z = pqr \end{cases}$ has unique solution.

Hence, by using Crammer's Rule or otherwise, solve (E) .

(c) If $p \neq 0$, solve (E) for the following cases:

(i) $p = q \neq r$

(ii) $p \neq q = r$

14. Consider the system of linear equations

$$(*)\begin{cases} (p-17)x + 4y + 3z = 5 \\ 2x + (p-10)y + 6z = 10 \\ 3x + 12y + (p-9)z = q \end{cases} \quad , \quad \text{where} \quad p, q \in \mathbb{R}.$$

(a) If (*) has unique solution,

(i) find the value(s) of p ;

(ii) solve for x, y and z in terms of p and q .

(b) If p = 0,

(i) find the value of q for (*) to be consistent;

(ii) solve the system of equations completely.

(c) If p = 18,

(i) find the value of q for (*) to be consistent;

(ii) solve the system of equations completely.